An Equilibrium Model of the Business Cycle

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This paper develops a theoretical example of a business cycle, that is, a model economy in which real output undergoes serially correlated movements about trend which are not explainable by movements in the availability of factors of production. The mechanism generating these movements involves unsystematic monetary-fiscal shocks, the effects of which are distributed through time due to information lags and an accelerator effect. Associated with these output movements are procyclical movements in prices, procyclical movements in the share of output devoted to investment, and, in a somewhat limited sense, procyclical movements in nominal rates of interest.

1. Introduction

This paper develops an exploratory business cycle theory in which unsystematic monetary shocks and an accelerator effect interact to generate serially correlated, "cyclical" movements in real output. Associated with these output movements are procyclical movements in prices, in the ratio of investment to output, and, in a rather special sense, in nominal interest rates. In contrast to conventional macroeconomic models, the model studied below has three distinguishing characteristics: prices and quantities at each point in time are determined in *competitive equilibrium;* the expectations of agents are *rational*, given the information available to them; information is *imperfect*, not only in the sense that the future is unknown, but also in the sense that no agent is perfectly informed as to the current state of the economy.

The attempt to discover a competitive equilibrium account of the business cycle may appear merely eccentric or, at best, an aesthetically

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motivated theoretical exercise. On the contrary, it is in fact motivated entirely by practical considerations. The problem of quantitatively assessing hypothetical countercyclical policies (say, a monetary growth rule or a fiscal stabilizer) involves imagining how agents will behave in a situation which has never been observed. To do this successfully, one must have some understanding of the way agents' decisions have been made in the past *and* some method of determining how these decisions would be altered by the hypothetical change in policy. Insofar as our descriptions of past behavior rely on arbitrary mechanical rules of thumb, adjustment rules, illusions, and unspecified institutional barriers, this task will be made difficult, or impossible. Who knows how "illusions" will be affected by an investment tax credit?[1]

In all of the models discussed in the paper, real output fluctuations are triggered by unanticipated monetary-fiscal shocks. The first theoretical task—indeed, the central theoretical problem of macroeconomics—is to find an analytical context in which this can occur and which does not at the same time imply the existence of persistent, recurrent, unexploited profit opportunities. Section 2 develops a neoclassical monetary growth model, with the aim of illustrating why this problem cannot be resolved within the class of aggregative models which view trade as taking place each period in a single, centralized market. This abstract environment, while analytically convenient, places too much information at the disposal of traders for cyclical behavior to be consistent with rationality.

In sections 3–5, this model is modified by viewing production and trade as occurring in a large number of markets which are imperfectly linked both physically and informationally. This is the analytical device first proposed by Phelps (1969) and since utilized by myself (1972, 1973), Lucas and Prescott (1974), and Barro (1975). As shown in Lucas (1972), this modification of the information structure of an otherwise neoclassical system leads to a real response to a purely nominal disturbance.

In Lucas (1972), and also in Sargent (1973*b*) and Sargent and Wallace (1973), however, these real movements are of no longer duration than the duration of the shock: no forces are present to account for the persistence or cumulation of the effects of the initial disturbance. In the present study, two such forces are introduced: information lags, such as to prevent even relevant past variables from becoming perfectly known, and physical capital, introducing a form of the familiar accelerator effect.

In the model set out in sections 3–5, agents' behavior is described by a pair of asset demand functions relating decisions to expected yields. That is, the link between tastes, technology, and demand behavior is not made explicit. On the other hand, the inference problem solved by agents to relate available information to expected yields is developed in some detail in section 6.

[1] This argument is much more fully developed in Lucas (1973*b*).

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Sections 7–9 develop certain conditions which an equilibrium solution must satisfy. The main novelty lies in the explication of the theoretical links between "structural" and "reduced-form" parameters implied by the rationality of agents' expectations formation: the "reduced form" depends on the "structure" for the usual reasons; and the "reduced form" determines the stochastic behavior of prices, and therefore affects the form of optimal forecast rules, and therefore the "structural equations" (decision rules).

Sections 10–12 describe the nature of the "cycle" produced by the model under three sets of assumptions on the parameters of the model. Section 10 exhibits the model of section 2 as a special case. Section 11 develops a purely monetary cycle in which capital plays no role. Section 12 describes a "monetary over investment" cycle.[2] It is the latter version which exhibits the qualitative characteristics cited in the first paragraph of this Introduction.

The cycles of sections 11 and 12 occur in a setting which abstracts from the existence of economy-wide securities markets. In view of the importance placed in the model on the partial nature of the information conveyed by the "local" prices at which agents trade, this abstraction may well be crucial. The informational role of economy-wide interest rates is briefly, if inconclusively, discussed in section 13.

Sections 14 and 15 discuss, briefly, some issues of testing and policy implications. Section 16 concludes the paper.

2. A Neoclassical Monetary Growth Model

Though the main concern of this paper is with oscillations of output and prices about a trend path, it will be useful to begin on more familiar ground with the discussion of a fairly standard, undisturbed neoclassical growth model. This will permit the fixing of notation and the early disposal of certain side issues.

Consider, to be specific, an economy producing a single output to be divided among private consumption, C_t, real government consumption, G_t, and next period's capital, K_{t+1}. The production function is f, and

$$C_t + G_t + K_{t+1} = f(K_t, N_t) + (1 - \delta)K_t \tag{2.1}$$

holds where N_t is employment. The function f has the usual monotonicity and curvature properties and is homogeneous of degree 1; δ is a depreciation rate.

There is a constant population of identical households which own all the factors of production. Labor is hired by firms at the wage W_t; capital is rented at U_t; and output is sold (to households and government) at P_t.

[2] The terminology is taken from Haberler's useful taxonomy of cycle theories (Haberler 1960).

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All three markets are competitive. Firms maximize current-period profit, so that in equilibrium

$$f_K(K_t, N_t) = \frac{U_t}{P_t}, \tag{2.2}$$

$$f_N(K_t, N_t) = \frac{W_t}{P_t}. \tag{2.3}$$

Households supply labor inelastically in quantity N, which fact, in conjunction with (2.2) and (2.3), determines equilibrium output, real wage, and real rental price, each as a function of K_t. In addition to owning the capital stock, households also hold a stock, M_t, of money balances and select an end-of-period balance, M_{t+1}.[3] Their budget constraint, given factor market equilibrium, is then

$$P_t(C_t + K_{t+1}) + M_{t+1} \leq P_t f(K_t, N) + P_t(1 - \delta)K_t + M_t. \tag{2.4}$$

The objective of the household is to maximize a subjectively discounted sum of current period utilities, where the latter depend on consumption and current holdings of real balances, M_{t+1}/P_t.

The model is completed by the specification of fiscal and money supply behavior. Let all of government consumption be financed by a monetary expansion at a constant, given rate μ. Then G_t is given implicitly by

$$P_t G_t = M_{t+1} - M_t = \mu M_t. \tag{2.5}$$

The dynamic behavior of the system will be determined once the demand on the part of households for the two forms of asset accumulation is specified. The most satisfactory way to do this, from some points of view, is to make explicit the household's preference functional and then to derive asset demands from households' infinite period maximum problem. An alternative route, taken here so as to set the stage for subsequent sections, is to postulate these demands directly.[4] Thus, let the demand for capital, K_{t+1}, depend on the expected one-period rates of return on capital and money, r_{kt} and r_{mt}, and the initial state of the household, K_t and M_t/P_t. The demand for money will depend on the same four variables. Given asset demands, consumption is implicit from (2.4).

[3] Here and in the remainder of the paper there will be only one nominal asset and it will be supplied exclusively by the government. I will call this asset "money" without qualifying this usage each time it arises, but it might as well be called "bonds" or "government liabilities." This means that the analysis will not be able to deal with questions involving the relative importance of monetary and fiscal disturbances, though it could without much difficulty be modified to do so.

[4] This is making virtue of analytical necessity, but there are definite intuitive advantages to a parametric, "certainty equivalent" approach as used here: the forecasting and choice problems solved by agents are separated (though we know this separation is artificial), and the distinct effects of each on decision rules are clearly seen (see n. 15 below).

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Again with an eye toward later developments, both relationships are assumed to be log linear. For the log of a variable, use the corresponding lowercase letter, so that k_t means the *log* of capital, and so forth. Then the second equality in (2.5) becomes [since $\log(1 + \mu) \cong \mu$]

$$m_{t+1} - m_t = \mu. \tag{2.6}$$

The two demand functions for assets are postulated as

$$k_{t+1} = \alpha_0 + \alpha_1 r_{kt} - \alpha_2 r_{mt} + \alpha_3 k_t, \tag{2.7}$$

$$m_{t+1} - p_t = \beta_0 - \beta_1 r_{kt} + \beta_2 r_{mt} + \beta_3 k_t. \tag{2.8}$$

The elasticities α_1, α_2, α_3 and β_1, β_2, β_3 are assumed to be positive; $\alpha_1 > \alpha_2$; $\beta_2 > \beta_1$; and α_3 and β_3 are less than unity. For completeness, the log of beginning-of-period real balances, $m_t - p_t$, should also appear on the right sides of (2.7) and (2.8) (since they figure in the budget constraint [2.4]). Here and in subsequent sections I shall neglect this "real balance effect" in order to focus on the effects of monetary changes on the two yields, r_{kt} and r_{mt}.[5]

The real one-period rate of return on capital is next period's real rental price, $f_K(K_{t+1}, N)$, less the depreciation rate. Approximating this by a linear function of the log of capital gives

$$r_{kt} = \delta_0 - \delta_1 k_{t+1}, \ \delta_1 > 0. \tag{2.9}$$

The rate of return on money is the percentage rate of deflation:

$$r_{mt} = p_t - p_{t+1}. \tag{2.10}$$

Since both of these rates of return depend on the values of future variables, both are "expectations" at the time they affect the decisions of traders. In the present context of certainty, it is natural to take these expectations to be correct (or rational). This assumption closes the system (2.6)–(2.10).

Substituting (2.6), (2.9), and (2.10) into (2.7) and (2.8), one obtains a pair of first-order difference equations in capital and real balances. The usual practice is to obtain the general solution to this system and then apply the boundary conditions that capital equal its historically given initial value and that real balances remain bounded and bounded away from zero as t tends to infinity. A slightly different method is adopted here, one which turns out to be more convenient when uncertainty is introduced.

Given the structure of the economy (the parameters α_i, β_j, δ_k, and μ), the pair (k_t, m_t) describes completely the *state* of the system at the beginning of period t. This leads naturally to *defining* a solution to be a set

[5] One can easily add a term "$\alpha_4(m_t - p_t)$" to the right of (2.7), and similarly to (2.8), and trace out the consequences. This leads to possibly interesting stability problems which are poorly understood (by me) and which I do not wish to confound with the cyclical complications which are introduced later.

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of functions relating equilibrium decisions and price to these two state variables. Since the system is linear, it is natural to conjecture the existence of solution functions of the form

$$k_{t+1} = \pi_{10} + \pi_{11}k_t + \pi_{12}m_t, \tag{2.11}$$

$$p_t = \pi_{20} + \pi_{21}k_t + \pi_{22}m_t. \tag{2.12}$$

Then *solving* means finding numbers $\pi_{10}, \ldots, \pi_{22}$ such that (2.6)–(2.12) hold *identically* in (k_t, m_t).

Substituting from (2.6) and (2.9)–(2.12) into (2.7) and (2.8) yields the required indentities in (k_t, m_t); equating the coefficients yields six equations in the unknown π_{ij}s:

$$\pi_{10} = \alpha_0 + \alpha_1\delta_0 - \alpha_1\delta_1\pi_{10} + \alpha_2\pi_{21}\pi_{10} + \alpha_2\pi_{22}\mu, \tag{2.13}$$

$$\pi_{11} = -\alpha_1\delta_1\pi_{11} - \alpha_2\pi_{21}(1 - \pi_{11}) + \alpha_3, \tag{2.14}$$

$$\pi_{12} = -\alpha_1\delta_1\pi_{12} + \alpha_2\pi_{21}\pi_{12}, \tag{2.15}$$

$$\mu - \pi_{20} = \beta_0 - \beta_1\delta_0 + \beta_1\delta_1\pi_{10} - \beta_2\pi_{21}\pi_{10} - \beta_2\pi_{22}, \tag{2.16}$$

$$-\pi_{21} = \beta_1\delta_1\pi_{11} + \beta_2\pi_{21}(1 - \pi_{11}) + \beta_3, \tag{2.17}$$

$$1 - \pi_{22} = \beta_1\delta_1\pi_{12} - \beta_2\pi_{21}\pi_{12}. \tag{2.18}$$

Equations (2.14) and (2.17) involve only π_{11} and π_{21}; their solution is diagrammed in Appendix A. As seen in figure A1, there are two solution pairs. One pair, with $\pi_{11} > 1$ and $\pi_{21} > 0$, has no economic significance and will be discarded.[6] The other is the desired solution; it satisfies

$$\frac{\alpha_3}{1 + \alpha_1\delta_1} < \pi_{11} < 1, \tag{2.19}$$

$$\pi_{21} < 0. \tag{2.20}$$

Inspection of (2.15) and (2.18) shows that one solution is

$$\pi_{12} = 0, \tag{2.21}$$

$$\pi_{22} = 1. \tag{2.22}$$

Since $\pi_{21} < 0$, there is no solution other than this classical one. Finally, the constants π_{10} and π_{20} are readily calculated from (2.13) and (2.16).

With these solution values, equation (2.11) is a stable first-order difference equation in capital stock. Capital tends monotonically to its stationary value, $\pi_{10}/(1 - \pi_{11})$. The behavior of prices is given by (2.12), given the paths of capital and money.

[6] This discarding of an unstable root is, of course, the step which is customarily "justified" by a transversality condition in models in which agents' maximum problems are made explicit (see, e.g., Brock 1973).

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Note first the sense in which money is "neutral" in this system. From (2.21) and (2.22), a once-and-for-all change in the *level* of money balances leads to a proportional change in the price level in the current and all future periods. There are no real effects. On the other hand, it is evident from (2.13) that changes in the *rate of increase* of money, μ, *will* have real consequences: the higher μ is the larger π_{10} is and hence the larger capital is all along its time path and at its stationary point. As Tobin (1965) and others have noted, this effect "works" through the real yield on money, which is, from (2.10) and (2.12),

$$r_{mt} = \pi_{21}(k_t - k_{t+1}) - \mu,$$

or, in the stationary state, simply the negative of the rate of monetary expansion.[7]

Note, second, the peripheral role played by the "flow variables"—output, private and government consumption, and employment—in determining the dynamic behavior of the system. The model is analyzed by first reducing it to the equations describing the motion of assets and their prices, solving these, and then returning to the determination of flow equilibrium. This characteristic, long familiar in more abstract theory, will carry over into subsequent sections. As a result, I will be discussing business cycles with scarcely a reference to such key magnitudes as employment, consumption, government spending, and real output. This may give an unfamiliar tone to much of what follows, but the translation back into the standard vocabulary is, I think, a straightforward exercise.

In particular, the reader may verify that the introduction of a taste for leisure and, consequently, a variable labor supply into the model of this section is easy to carry out, with no effect on the form of (2.7) and (2.8). This modification is obviously essential for business cycle theory and will be taken for granted below.

Finally, and in sharp contrast to traditional macroeconomic models, the solution found above remains valid under very wide variations in what is assumed about the behavior of money. To take one example, suppose $m_{t+1} - m_t$ is a sequence of independent, normal variates, each with mean μ and variance σ^2. If (2.9) and (2.10) are reinterpreted as *expected* rates of return, conditional on information available up through t, then (2.11) and (2.12) remain a solution, with the *same* coefficients π_{ij} as found above. In view of the emphasis often put on the distinction between anticipated and unanticipated monetary changes, this fact may seem

[7] This nonneutrality of inflation did not appear in Lucas (1972) or Sargent (1973*b*), since both papers excluded capital formation. This led Tobin (1973), and perhaps others, to wonder how monetary distortions present in models with certainty and perfect foresight can disappear when uncertainty is introduced. The answer is, they do not. The point of Lucas (1972) and Sargent (1973*b*) is not that the introduction of uncertainty removes long-familiar neoclassical nonneutralities but, rather, that it does not in itself introduce new ones.

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paradoxical. It results from the fact that in a competitive market the current price is part of traders' information sets. Thus, a trader who knows the coefficients of (2.12) and the current real capital k_t *knows* m_t prior to committing himself, regardless of whether it is announced or not, or anticipated or not.

3. A Cycle Model: Introduction

The above discussion of a monetary growth model concluded with the observation that merely introducing "noise" into monetary policy was not sufficient to induce the sort of responses in real and nominal variables which occur during the observed business cycle. The problem is that in an economy in which all trading occurs in a single competitive market, there is "too much" information in the hands of traders for them ever to be "fooled" into altering real decision variables.

To get away from this analytical difficulty, but not so far away as to preclude a simple description of aggregate behavior, I shall adopt the device proposed by Phelps (1969) and, since utilized in similar contexts by Lucas (1972, 1973) and Lucas and Prescott (1974), of thinking of trading as occurring in distinct markets, or "islands." Such a system is described in this section and analyzed in the remainder of the paper.

At the beginning of a period, traders are distributed in some way over a continuum of markets. Each market has capital in place, as determined by the preceding period's trading. There is a stock of money in the hands of traders; in addition, government purchases introduce new money in a way which varies stochastically from market to market and period to period. Within each market, production, exchange, and asset accumulation take place exactly as described in the preceding section, with the sole difference being that the two yields, r_{kt} and r_{mt}, are conditional expectations rather than known numbers. Once trading is complete, agents select a new market at random, new monetary shocks are realized, and the process continues.[8]

Capital accumulated in a particular market is assumed to remain there into the next period, though its owners move on. The dollar return to capital is then received by shareholders after trading is complete. The size of this one-period "float" is taken to be proportional to the stock of money (though, in fact, this cannot hold exactly) and is neglected in what follows.

[8] The idea behind this island abstraction is not, of course, to gain insight into maritime affairs, or to comment on the aimlessness of life. It is intended simply to capture in a tractable way the fact that economic activity offers agents a succession of ambiguous, unanticipated opportunities which cannot be expected to stay fixed while more information is collected. It seems safe and, for my purposes, sensible to abstract here from the fact that in reality this situation can be slightly mitigated by the purchase of additional information.

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The financing of investment is entirely "internal": there are no economy-wide markets for capital funds.[9]

All exchange in this economy takes place at competitive market clearing prices. The behavior of each trader is *rational* both in the conventional sense of optimal, given objectives and expectations, and in the Muthian sense (Muth 1961) that available information is optimally utilized in forming expectations. In order that the latter assumption have an operational meaning, the analysis will be restricted to the situation in which the relevant distributions have settled down to stationary values and can thus be "known" by traders.

The central economic ingredients of this model will, as in the preceding section, be the asset demand functions (2.7) and (2.8), which will now differ from market to market due to variations in capital stock and in information. The aim of the analysis will, also as above, be to obtain the analogue to the solutions (2.11) and (2.12) for the motion of the state variables and their relative price. The major difference induced by the introduction of relative and aggregate "noise" will be in the calculation, by agents, of the expected yields r_{kt} and r_{mt}, which will now be mean values conditioned on limited information rather than perfectly foreseen realizations.

It will be convenient to develop these elements in the reverse of the usual order. In the next section, the information structure of the economy is described and the solution of the model is stated formally. Next, in section 5, the asset demand functions are restated and the two expected yields redefined. The inference problem solved by agents is treated in section 6, completing the statement of the model.

4. Notation and a Formal Solution

To move toward an explicit description of the economy described above, think of trade as occurring in a continuum of separated markets z, $0 < z < 1$, where z is an index of location. The system is driven by stochastic injections of new money (in the form of governmental spending) which vary over time and over markets at a given time. Let the average (over markets) percentage increase in money be x_t, where $x_t \sim N(\mu, \sigma^2)$. Market z receives an increment which deviates from the average by the percentage amount $\theta_t(z)$, where

$$\theta_t(z) = \rho\theta_{t-1}(z) + \varepsilon_t(z),\ 0 < \rho < 1 \tag{4.1}$$

and $\varepsilon_t(z) \sim N(0, \sigma_\varepsilon^2)$. Take $\varepsilon_t(z)$ and x_s to be independent for all s, t, z and $\varepsilon_t(z)$ and $\varepsilon_s(z')$ to be independent, unless $s = t$ and $z = z'$. Then

[9] See sec. 13 for a discussion of the probable effects of introducing an economy-wide bond market.

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the stationary distribution of $[x_t, \theta_t(z)]$ for any fixed location z will be normal with mean $(\mu, 0)$ and covariance matrix

$$\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma_\theta^2 \end{pmatrix},$$

where $\sigma_\theta^2 = \sigma_\varepsilon^2/(1 - \rho^2)$. None of the shocks $\varepsilon_t(z)$, $\theta_t(z)$, and x_t are ever observed by agents, but their distributions are taken to be constant and known by agents.[10]

As a consequence of these shocks, the only ones affecting the economy, capital stock may be expected to vary over time and across markets. Let $k_t(z)$ denote the log of beginning-of-period capital in z at t and let k_t be the average value of $k_t(z)$ over all markets. Use $u_t(z) = k_t(z) - k_t$ to denote the deviation from average of market z's capital. These three variables will follow a stochastic process to be determined. Denote the stationary distribution of $u_t(z)$ as $N(0, \sigma_u^2)$. One would expect the persistent, relative shocks $\theta_t(z)$ to affect capital movements, so that $\sigma_{u\theta} = E[\theta_t(z)u_t(z)]$ will be nonzero. These distributional facts are also assumed known to agents, though $k_t(z)$ and k_t cannot be directly observed.

Also as a consequence of the disturbances, individuals in different markets will acquire differing amounts of money during a trading period. Think of a large number of agents, each selecting next period's market at random, so that the distribution of agents by their money balances will be the same in all markets. In the normal, log-linear structure to be used below, the only changing feature of this distribution will be its logarithmic mean, denoted (as in sec. 2) by m_t. This average follows the random walk

$$m_{t+1} = m_t + x_t. \tag{4.2}$$

Assume that m_t is not directly observed by agents. Equations (4.1) and (4.2) together give a complete description of the flows of money through the various markets in this economy and all the relevant information on the distribution of money among agents.

According to the above description, then, the aggregate (or average) state of the economy is described by the values k_t, m_t, and x_t of capital stock, money, and nominal government spending. The situation of an individual market z is described by its capital relative to average, $u_t(z) = k_t(z) - k_t$, and the government spending it receives relative to average, $\theta_t(z)$.

As agents diffuse through this system, they observe none of these variables directly. Each period, however, they trade goods for money at a market clearing price $p_t(z)$. The history of prices $p_t(z)$, $p_{t-1}(z')$,

[10] The assumption that these unobserved distributions are "known" need not be taken as a literal description of the way agents think of their environment. It is just a convenient way of assuming that agents use the data available to them in the best possible way.

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$p_{t-2}(z'')$, . . . , observed by an individual is his source of information on the current state of the economy and of the market z in which he currently finds himself; equivalently, this history is his source of information on future prices.[11] Since traders follow different paths, each will have different information in hand, so that in general one would need to describe the informational state of the economy by a distribution of agents by information held. To complicate matters still further, this informational state will influence prices and will then itself be an object of speculation—agents will form expectations about the expectations of others. Two further conventions will help to simplify this complex picture. First, assume that each agent summarizes the price history $(p_{t-1}, p_{t-2}, \ldots)$ observed by him in a pair $(\hat{k}_t, \hat{m}_t)$, his unbiased estimate of the current values of the aggregate state variables, (k_t, m_t).[12] Second, prior to trading each period, these estimates are "pooled" by traders by simple averaging, so that a single pair $(\hat{k}_t, \hat{m}_t)$ of numbers describes the perceptions of *all* agents. Let these perceptions be normally distributed about the actual aggregate state, with the covariance matrix

$$\begin{pmatrix} \sigma_k^2 & \sigma_{mk} \\ \sigma_{mk} & \sigma_m^2 \end{pmatrix}.$$

The *state* of a particular market z, then, is fully described by seven numbers: $\hat{k}_t$, $\hat{m}_t$, k_t, m_t, x_t, $\theta_t(z)$, $u_t(z)$. Agents do not know this state, though of course they do know their own expectations $(\hat{k}_t, \hat{m}_t)$. On the basis of the latter, they have a well-formed opinion of the relevant variables they cannot observe: k_t, m_t, x_t, $\theta_t(z)$, $u_t(z)$. Specifically, they believe (correctly) that this random vector is normally distributed with a mean $(\hat{k}_t, \hat{m}_t, \mu, 0, 0)$ and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_k^2 & \sigma_{km} & 0 & 0 & 0 \\ \sigma_{km} & \sigma_m^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_\theta^2 & \sigma_{u\theta} \\ 0 & 0 & 0 & \sigma_{u\theta}^2 & \sigma_u^2 \end{bmatrix}. \tag{4.3}$$

This completes the description of both the actual state of the economy and the opinions agents have as to this state.

As in section 2, the aim of the analysis will be to define and study the equilibrium motion of this system from state to state. Also as in section 2, one has the choice of thinking of equilibrium as a set of *time paths* of assets and prices, or as a set of *functions* which specify prices and asset

[11] This neglects, as Sargent has pointed out to me, the information conveyed to traders when they receive the dividend "check" for their investment of two periods earlier.

[12] See n. 10 above.

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movements, given the current state. Taking the latter route, let an equilibrium take the form[13]

$$k_{t+1}(z) = \pi_{10} + \pi_{11}\hat{k}_t + \pi_{12}\hat{m}_t + \pi_{13}[k_t + u_t(z)] + \pi_{14}m_t + \pi_{15}[x_t + \theta_t(z)], \tag{4.4}$$

$$p_t(z) = \pi_{20} + \pi_{21}\hat{k}_t + \pi_{22}\hat{m}_t + \pi_{23}[k_t + u_t(z)] + \pi_{24}m_t + \pi_{25}[x_t + \theta_t(z)]. \tag{4.5}$$

Subsequent sections will be devoted first to developing a set of conditions which these coefficients π_{ij} must satisfy and then to developing the implications of these conditions.

5. Asset Demand Functions

Current-period flow equilibrium is determined in each market exactly as in section 2. I shall focus, then, on the asset demand functions (2.7) and (2.8), repeated here in a notation which emphasizes their market specificity but is otherwise unchanged:

$$k_{t+1}(z) = \alpha_0 + \alpha_1 r_{kt}(z) - \alpha_2 r_{mt}(z) + \alpha_3 k_t(z), \tag{5.1}$$

$$m_t^d(z) - p_t(z) = \beta_0 - \beta_1 r_{kt}(z) + \beta_2 r_{mt}(z) + \beta_3 k_t(z). \tag{5.2}$$

The parameters α_i, β_j are restricted as in section 2. In addition to (5.1) and (5.2), money supply follows (4.1) and (4.2), and

$$m_t^d(z) = m_t + x_t + \theta_t(z) \tag{5.3}$$

holds.

The two asset yields, r_{kt} and r_{mt}, are conceptually as in section 2 but in the present case of uncertainty will be taken to be conditional means. The return on money is, again, the expected deflation rate. Since traders will choose next period's market at random, the expected rate relevant in z at t is

$$r_{mt}(z) = p_t(z) - \bar{p}_{t+1}^e(z), \tag{5.4}$$

where $p_t(z)$ is the observed current price and $\bar{p}_{t+1}^e(z)$ is the expected value of next period's *average* price level, conditional on information available in z at t.

The return on capital is, as before, the expected real rental price. Since capital accumulated in z remains there into the next period, the nominal rental will be proportional to the *local* price prevailing next

[13] With a modest application of intuition, one can specify some of the solution parameters in advance. (E.g., since two markets with the same total $m_t + x_t + \theta$ should look alike, one should have $\pi_{14} = \pi_{15}$ and $\pi_{24} = \pi_{25}$.) There is no harm in carrying along extra parameters, however, and since intuitions differ, one may as well develop such facts formally. This is done in sec. 9.

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period. On the other hand, since dividends will be spent elsewhere, the appropriate deflator is an expected *average* price. In addition to these price effects, the dampening effect of diminishing returns will also, as in section 2, be present. In view of the peripheral role of diminishing returns over the cycle, I shall neglect the latter effect here and write

$$r_{kt}(z) = p^e_{t+1}(z) - \bar{p}^e_{t+1}(z), \tag{5.5}$$

where $p^e_{t+1}(z)$ is the price expected to prevail locally, next period, on the basis of current-period information.

6. The Formation of Expectations

Since the rates of return which figure in the demand functions, (5.1) and (5.2), are not directly observable, they (or the expected prices which comprise them) must be inferred by agents from available information. This inference problem is the subject of this section.

The information sets and "priors" of agents are described in section 4. Agents know the coefficients of the solution (4.4)–(4.5) and take the joint distribution of the state vector $[k_t, m_t, x_t, \theta_t(z), u_t(z)]$ to be normal, with mean $(\hat{k}_t, \hat{m}_t, \mu, 0, 0)$ and covariance matrix $\sum$ as given by (4.3). Then, prior to trading, they observe the equilibrium price, which as a function of the unobserved state vector carries additional information. On the basis of this new information, agents form a posterior distribution on the state vector to be used in forecasting. Denote the mean of this posterior distribution $[\tilde{k}_t, \tilde{m}_t, \tilde{x}_t, \tilde{\theta}_t(z), \tilde{u}_t(z)]$.

From (4.5), the price which, prior to trading, had been expected to prevail was

$$\hat{p}_t = \pi_{20} + \pi_{21}\hat{k}_t + \pi_{22}\hat{m}_t + \pi_{23}\hat{k}_t + \pi_{24}\hat{m}_t + \pi_{25}\mu.$$

Also from (4.5), the price which in fact prevails is

$$\begin{aligned} p_t(z) = \hat{p}_t &+ \pi_{23}(k_t - \hat{k}_t) + \pi_{23}u_t(z) \\ &+ \pi_{24}(m_t - \hat{m}_t) + \pi_{25}(x_t - \mu) + \pi_{25}\theta_t(z). \end{aligned} \tag{6.1}$$

Thus, $[k_t, m_t, x_t, \theta_t(z), u_t(z)]$ and $p_t(z) - \hat{p}_t$ are, from the point of view of agents, jointly normally distributed variates with a covariance matrix given by $\sum$ and (6.1). A straightforward calculation yields the conditional means[14]

$$\tilde{k}_t = \hat{k}_t + \sigma_p^{-2}(\pi_{23}\sigma_k^2 + \pi_{24}\sigma_{mk})[p_t(z) - \hat{p}_t], \tag{6.2}$$

$$\tilde{m}_t = \hat{m}_t + \sigma_p^{-2}(\pi_{23}\sigma_{mk} + \pi_{24}\sigma_m^2)[p_t(z) - \hat{p}_t], \tag{6.3}$$

$$\tilde{x}_t = \mu + \sigma_p^{-2}\pi_{25}\sigma^2[p_t(z) - \hat{p}_t], \tag{6.4}$$

$$\tilde{\theta}_t(z) = \sigma_p^{-2}(\pi_{23}\sigma_{u\theta} + \pi_{25}\sigma_\theta^2)[p_t(z) - \hat{p}_t], \tag{6.5}$$

$$\tilde{u}_t(z) = \sigma_p^{-2}(\pi_{23}\sigma_u^2 + \pi_{25}\sigma_{u\theta})[p_t(z) - \hat{p}_t], \tag{6.6}$$

[14] See, e.g., Graybill (1961, theorem 3.10, p. 63).

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where

$$\sigma_p^2 = \pi_{23}^2\sigma_k^2 + 2\pi_{23}\pi_{24}\sigma_{mk} + \pi_{24}^2\sigma_m^2 + \pi_{25}^2\sigma^2 + \pi_{23}^2\sigma_u^2 + 2\pi_{23}\pi_{25}\sigma_{u\theta} + \pi_{25}^2\sigma_\theta^2 \tag{6.7}$$

is the variance of actual price about its prior mean.

One notes that each posterior (conditional) mean is simply the prior mean, corrected by a term which incorporates the new information contained in the market price, $p_t(z) - \hat{p}_t$. In each case the weight attached to the new information $p_t(z) - \hat{p}_t$ in (6.2)–(6.6) is the simple regression coefficient of the shock in question on $p_t(z) - \hat{p}_t$. Thus, for example, in (6.4), $\sigma_p^{-2}\pi_{25}\sigma^2$ is the covariance of $p_t(z) - \hat{p}_t$ and $x_t - \mu$ divided by the variance of price.

The estimates (6.2)–(6.6) are now used by traders both to update their estimates $(\hat{k}_t, \hat{m}_t)$ of the aggregate state of the economy and to form unbiased expectations $r_{kt}(z)$ and $r_{mt}(z)$ of the yields which are relevant to the asset demand decision. Using $E_z(\cdot)$ to denote an expectation formed in z in t, one has, from (4.4),

$$\begin{aligned}\hat{k}_{t+1}(z) &= E_z(\pi_{10} + \pi_{11}\hat{k}_t + \pi_{12}\hat{m}_t + \pi_{13}k_t + \pi_{14}m_t + \pi_{15}x_t) \\ &= \pi_{10} + \pi_{11}\hat{k}_t + \pi_{12}\hat{m}_t + \pi_{13}\tilde{k}_t + \pi_{14}\tilde{m}_t + \pi_{15}\tilde{x}_t,\end{aligned}$$

where $\hat{k}_{t+1}(z)$ is the posterior estimate of k_{t+1} based on market z information. Now substitute from (6.2), (6.3), (6.4), and (6.1) and average over markets z to obtain the *average* estimate of k_{t+1}:

$$\begin{aligned}\hat{k}_{t+1} &= \pi_{10} + (\pi_{11} + \pi_{13})\hat{k}_t + (\pi_{12} + \pi_{14})\hat{m}_t + \pi_{15}\mu \\ &\quad + B_1[\pi_{23}(k_t - \hat{k}_t) + \pi_{24}(m_t - \hat{m}_t) + \pi_{25}(x_t - \mu)],\end{aligned} \tag{6.8}$$

where B_1 is a function of the elements of $\sum$ and the π_{ij}, given for reference in Appendix B. Similar calculations give

$$\begin{aligned}\hat{m}_{t+1} &= \hat{m}_t + \mu + B_2[\pi_{23}(k_t - \hat{k}_t) \\ &\quad + \pi_{24}(m_t - \hat{m}_t) + \pi_{25}(x_t - \mu)],\end{aligned} \tag{6.9}$$

where B_2 is given in Appendix B.

The expected yields as defined by (5.4) and (5.5) are calculated in the same way. For example,

$$\begin{aligned}r_{mt}(z) &= p_t(z) - \tilde{p}_{t+1}^e(z) \\ &= \pi_{20} + \pi_{21}\hat{k}_t + \pi_{22}\hat{m}_t + \pi_{23}[k_t + u_t(z)] \\ &\quad + \pi_{24}m_t + \pi_{25}[x_t + \theta_t(z)] \\ &\quad - E_z(\pi_{20} + \pi_{21}\hat{k}_{t+1} + \pi_{22}\hat{m}_{t+1} + \pi_{23}k_{t+1} \\ &\quad + \pi_{24}m_{t+1} + \pi_{25}x_{t+1}),\end{aligned}$$

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using the solution for price, (4.5). Observing that $E_z[\hat{k}_{t+1}] = E_z[k_{t+1}]$ and $E_z[\hat{m}_{t+1}] = E_z[m_{t+1}]$ and using the solution for capital (4.4) and the monetary rule (4.2), one finds

$$\begin{aligned} r_{mt}(z) = {} & \pi_{21}\hat{k}_t + \pi_{22}\hat{m}_t + \pi_{23}[k_t + u_t(z)] + \pi_{24}m_t + \pi_{25}[x_t + \theta_t(z)] \\ & - (\pi_{21} + \pi_{23})E_z(\pi_{10} + \pi_{11}\hat{k}_t + \pi_{12}\hat{m}_t + \pi_{13}k_t \\ & \quad + \pi_{14}m_t + \pi_{15}x_t) \\ & - (\pi_{22} + \pi_{24})E_z(m_t + x_t) \\ & - \pi_{25}\mu. \end{aligned}$$

Now using the estimates (6.3)–(6.5) and (6.1) and collecting terms, one finds

$$\begin{aligned} r_{mt}(z) = {} & (\pi_{21} + \pi_{23})[1 - (\pi_{11} + \pi_{13})]\hat{k}_t \\ & - (\pi_{21} + \pi_{23})(\pi_{12} + \pi_{14})\hat{m}_t \\ & + (1 - A_1)[\pi_{23}(k_t - \hat{k}_t) + \pi_{23}u_t(z) + \pi_{24}(m_t - \hat{m}_t) \\ & \quad + \pi_{25}(x_t - \mu) + \pi_{25}\theta_t(z)] \\ & + C_1, \end{aligned} \tag{6.10}$$

where A_1 is given in Appendix B and C_1 is a constant which will be ignored in the sequel. An analogous calculation gives an expression for the expected yield on capital:

$$\begin{aligned} r_{kt}(z) = {} & A_2[\pi_{23}(k - \hat{k}_t) + \pi_{23}u_t(z) + \pi_{24}(m_t - \hat{m}_t) \\ & + \pi_{25}(x_t - \mu) + \pi_{25}\theta_t(z)], \end{aligned} \tag{6.11}$$

where A_2 is given in Appendix B.

This completes the statement of the model, though a mathematical definition of its solution is still two sections away. The given economic parameters are the coefficients in the asset demand functions, $\alpha_0, \ldots, \alpha_3$ and $\beta_0, \ldots, \beta_3$, the parameter ρ, and the two variances σ^2 and σ_ε^2. The economic assumptions imply a set of conditions relating these parameters to the solution parameters: the coefficients π_{ij} in (4.4) and (4.5) and the remaining elements of the covariance matrix $\sum$. Implications on the slope coefficients will be developed in the following section; those on the covariance matrix in section 8.

7. Implications on Slope Coefficients

Inserting the expressions for expected yields given by (6.10) and (6.11) into the capital demand function (5.1) yields $k_{t+1}(z)$ as a linear function of the current state variables in market z. A second expression of this

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functional relationship is given by (4.4). Since these two relationships are equivalent, their right-hand sides must be identically equal in $\hat{k}_t$, $\hat{m}_t$, k_t, m_t, x_t, $\theta_t(z)$, and $u_t(z)$. Equating coefficients gives five conditions:

$$\pi_{11} = -[\alpha_1 A_2 - \alpha_2(1 - A_1)]\pi_{23} - \alpha_2(\pi_{21} + \pi_{23})(1 - \pi_{11} - \pi_{13}), \tag{7.1}$$

$$\pi_{12} = -[\alpha_1 A_2 - \alpha_2(1 - A_1)]\pi_{24} + \alpha_2(\pi_{21} + \pi_{23})(\pi_{12} + \pi_{14}), \tag{7.2}$$

$$\pi_{13} = [\alpha_1 A_2 - \alpha_2(1 - A_1)]\pi_{23} + \alpha_3, \tag{7.3}$$

$$\pi_{14} = [\alpha_1 A_2 - \alpha_2(1 - A_1)]\pi_{24}, \tag{7.4}$$

$$\pi_{15} = [\alpha_1 A_2 - \alpha_2(1 - A_1)]\pi_{25}. \tag{7.5}$$

Equating money demand and supply (eliminating $m_t^d(z)$ between [5.2] and [5.3]) and inserting the yields (6.10) and (6.11) give an expression for the current price, $p_t(z)$. Since the expression (4.5) must be equivalent, one obtains five more conditions:

$$\pi_{21} = [\beta_1 A_2 - \beta_2(1 - A_1)]\pi_{23} - \beta_2(\pi_{21} + \pi_{23})(1 - \pi_{11} - \pi_{13}), \tag{7.6}$$

$$\pi_{22} = -[\beta_1 A_2 - \beta_2(1 - A_1)]\pi_{24} + \beta_2(\pi_{21} + \pi_{23})(\pi_{12} + \pi_{14}), \tag{7.7}$$

$$\pi_{23} = [\beta_1 A_2 - \beta_2(1 - A_1)]\pi_{23} - \beta_3, \tag{7.8}$$

$$\pi_{24} = [\beta_1 A_2 - \beta_2(1 - A_1)]\pi_{24} + 1, \tag{7.9}$$

$$\pi_{25} = [\beta_1 A_2 - \beta_2(1 - A_1)]\pi_{25} + 1. \tag{7.10}$$

The two additional conditions for the constant terms π_{10} and π_{20} will be neglected.

So far, then, we have ten equations involving the ten unknown π_{ij} *and* (via A_1 and A_2) the five unknown elements of $\sum$: σ_k^2, σ_m^2, σ_{mk}, $\sigma_{u\theta}$, and σ_u^2.

8. Implications on Covariances

Rationality of expectations also implies that the covariance matrix $\sum$ used by agents in forecasting is at the same time the *true* stationary covariance matrix. For the exogenously given moments σ^2 and σ_θ^2, this holds by direct assumption. For the other elements of $\sum$, some calculations are involved.

From (4.4) one observes that

$$u_{t+1}(z) = k_{t+1}(z) - k_t = \pi_{13} u_t(z) + \pi_{15}\theta_t(z). \tag{8.1}$$

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Then, using the fact that

$$\theta_{t+1}(z) = \rho\theta_t(z) + \varepsilon_t,$$

the stationary moments σ_u^2 and $\sigma_{u\theta}$ are given by

$$\sigma_u^2 = \frac{\pi_{15}^2}{1 - \pi_{13}^2} \frac{1 + \rho\pi_{13}}{1 - \rho\pi_{13}} \sigma_\theta^2, \tag{8.2}$$

$$\sigma_{u\theta} = \frac{\rho\pi_{15}}{1 - \pi_{13}} \sigma_\theta^2, \tag{8.3}$$

provided $|\pi_{13}| < 1$.

Averaging both sides of the solution (4.4) with respect to z gives an expression for k_{t+1}. Subtracting this equation from (6.8), one obtains

$$\hat{k}_{t+1} - k_{t+1} = (\pi_{13} - \pi_{23}B_1)(\hat{k}_t - k_t) + (\pi_{14} - \pi_{24}B_1)(\hat{m}_t - m_t) - (1 - \pi_{25}B_2)(x_t - \mu), \tag{8.4}$$

$$\hat{m}_{t+1} - m_{t+1} = -\pi_{21}B_2(\hat{k}_t - k_t) + (1 - \pi_{24}B_2)(\hat{m}_t - m_t) - (1 - \pi_{25}B_2)(x_t - \mu). \tag{8.5}$$

Provided the deterministic part of the pair (8.4)–(8.5) (that is, the system obtained by setting $x_t = \mu$ for all t) is stable, it is a familiar calculation to obtain three linear equations in the moments σ_k^2, σ_m^2, and σ_{mk}. For reference, write it as

$$\begin{bmatrix} \sigma_k^2 \\ \sigma_{mk} \\ \sigma_m^2 \end{bmatrix} = K_1 \begin{bmatrix} \sigma_k^2 \\ \sigma_{mk} \\ \sigma_m^2 + \sigma^2 \end{bmatrix}, \tag{8.6}$$

where the 3×3 matrix K_1 is written in Appendix B.

9. Mathematical Solution: Preliminaries

The mathematical problem is now sharpened to: for given $\alpha_1, \ldots, \alpha_3$, $\beta_1, \ldots, \beta_3$, σ^2, σ_ε^2, and ρ, find $\pi_{11}, \ldots, \pi_{15}, \ldots, \pi_{21}, \ldots, \pi_{25}$, σ_u^2, $\sigma_{u\theta}$, σ_k^2, σ_m^2, and σ_{mk} which satisfy (7.1)–(7.10), (8.2), (8.3), and (8.6) such that the difference equations (8.1), (8.4), and (8.5) are stable. In this section, the size of the system will be drastically reduced by solving for some coefficients in terms of others.

First, adding (7.1) to (7.3) and (7.6) to (7.8) gives two equations in the sums $\pi_{11} + \pi_{13}$ and $\pi_{21} + \pi_{23}$. These are essentially the same equations solved for π_{11} and π_{21} in section 2; their solution is diagrammed in Appendix A. As in section 2, there are two solution pairs, one of which is of economic interest. Denote these solution values $\pi_1 = \pi_{11} + \pi_{13}$ and

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$-\pi_2 = \pi_{21} + \pi_{23}$. From figure A1 (Appendix A) one sees that these solutions satisfy

$$\alpha_3 < \pi_1 < 1 \tag{9.1}$$

and

$$\frac{\beta_3}{1 + \beta_2} < \pi_2 < \beta_3. \tag{9.2}$$

Next, add (7.2) and (7.4) and conclude that

$$\pi_{12} + \pi_{14} = 0. \tag{9.3}$$

Similarly, add (7.7) and (7.9) and conclude that

$$\pi_{22} + \pi_{24} = 1. \tag{9.4}$$

Neither of these classical neutrality-of-money results should come as a surprise.

Third, from (7.9) and (7.10), conclude that

$$\pi_{25} = \pi_{24} \tag{9.5}$$

and from (7.4), (7.5), and (9.5) that

$$\pi_{15} = \pi_{14}. \tag{9.6}$$

Fourth, solving (7.8) and (7.9) for π_{23} gives

$$\pi_{23} = -\beta_3\pi_{24}. \tag{9.7}$$

Then, from (7.3), (7.4), and (9.7), one obtains

$$\pi_{13} = \alpha_3 - \beta_3\pi_{14}. \tag{9.8}$$

Reviewing the facts stated in (9.1)–(9.8), one sees that all slope coefficients π_{ij} have been expressed in terms of π_{14} and π_{24} and the now "known" numbers π_1 and π_2. Let me rename π_{14} and π_{24}, π_3 and π_4 respectively. In terms of these parameters $\pi_1, \ldots, \pi_4$, (8.4) and (8.5) become

$$\begin{aligned}\hat{k}_{t+1} - k_{t+1} = {} & (\alpha_3 - \beta_3\pi_3 + \beta_3\pi_4 B_1)(\hat{k}_t - k_t) \\ & + (\pi_3 - \pi_4 B_1)(\hat{m}_t - m_t) - (\pi_3 - \pi_4 B_1)(x_t - \mu),\end{aligned} \tag{9.9}$$

$$\begin{aligned}\hat{m}_{t+1} - m_{t+1} = {} & \beta_3\pi_4 B_2(\hat{k}_t - k_t) \\ & + (1 - \pi_4 B_2)(\hat{m}_t - m_t) - (1 - \pi_4 B_2)(x_t - \mu).\end{aligned} \tag{9.10}$$

The motion of aggregate capital and the price level, k_t and p_t, is from (4.4) and (4.5):

$$\begin{aligned}k_{t+1} = {} & \pi_{10} + \pi_1 k_t + (\pi_1 - \alpha_3 + \beta_3\pi_3)(\hat{k}_t - k_t) \\ & - \pi_3(\hat{m}_t - m_t) + \pi_3 x_t,\end{aligned} \tag{9.11}$$

$$\begin{aligned}p_t = {} & \pi_{20} + m_t - \pi_2 k_t + (\beta_3\pi_4 - \pi_2)(\hat{k}_t - k_t) \\ & + (1 - \pi_4)(\hat{m}_t - m_t) + \pi_4 x_t.\end{aligned} \tag{9.12}$$

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The information contained in the 10 equations (7.1)–(7.10) may now be conveniently restated as

$$\pi_3 = [\alpha_1 A_2 - \alpha_2(1 - A_1)]\pi_4, \tag{9.13}$$

$$\pi_4 = [\beta_1 A_2 - \beta_2(1 - A_1)]\pi_4 + 1. \tag{9.14}$$

The terms A_1, A_2, B_1, and B_2 may similarly be expressed in terms of π_1, π_2, π_3, and π_4; these simplified expressions are given in Appendix B.

The problem of solving for the equilibrium parameter values is now reduced to: find π_3, π_4, σ_u^2, $\sigma_{u\theta}$, σ_m^2, σ_{mk}, and σ_k^2 such that (9.13), (9.14), (8.2), (8.3), and (8.6) are satisfied. The fact that the covariance structure and the response coefficients π_3 and π_4 are mutually dependent makes this task difficult, and results have been obtained for special cases only. These will be discussed in detail in subsequent sections.

At this point, however, the general nature of the dynamic system is fairly clear. Equations (9.9) and (9.10) describe the consequences of the unsystematic shocks, x_t, on the deviations between the perceived and the actual aggregate state of the economy, $\hat{k}_t - k_t$ and $\hat{m}_t - m_t$. This autonomous, two-equation system converts a one-time pulse of monetary "misinformation" into an extended, distributed lag effect. Equation (9.11) describes the motion of capital stock as the sum of a "deterministic" part, which is essentially the same as the capital path found in section 2, and autocorrelated deviations about this path, determined by the shocks and their lagged effects from (9.9) and (9.10). The effects on price are given in (9.12).

10. Case 1: Centralized Market Clearing

The case in which the relative demand variance σ_ε^2 is zero corresponds exactly to the situation discussed briefly at the end of section 2 in which monetary shocks are the only exogenous disturbance to which the economy is subject. Since with no variation in θ all markets are identical, an economy with $\sigma_\varepsilon^2 = 0$ may be viewed as one in which all trade takes place in a single market.

The algebra appropriate to this case is given in Appendix C. Briefly, one observes first that the function A_2 is zero when $\sigma_\varepsilon^2 = 0$, implying from (6.11) that expected real yields on capital do not change with monetary shocks. It follows that $\sigma_k^2 = \sigma_{mk} = \sigma_m^2 = 0$ is the solution to (8.6). Then, from (9.13) and (9.14), the coefficients π_3 and π_4 are 0 and 1, respectively.

Inserting these values into (9.9) and (9.10) gives the solution for equilibrium price and capital accumulation. In this case, $k_t(z) = k_t = \hat{k}_t$ for all markets and all periods. Similarly, $m_t = \hat{m}_t$. In short, there is no misinformation. The effect of monetary changes on capital is nil ($\pi_3 = 0$);

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there is a proportional effect on nominal prices ($\pi_4 = 1$). Monetary changes are accurately conveyed to agents via price movements, even though unanticipated, and the response is simply an adjustment in nominal units.

While this case is of no particular interest substantively, it does serve to "justify" the apparatus set up in preceding sections, to which I shall shortly return. The introduction of separate, informationally distinct markets is not a step toward "realism" or (obviously) "elegance" but, rather, an analytical departure which appears essential (in some form) to an explanation of the way in which business cycles can arise and persist in a competitive economy.

11. Case 2: A Purely Monetary Cycle[15]

The case in which capital stock does not respond to monetary shocks may, in contrast to the preceding case, be of practical importance, since cyclical variations in capital appear, at least at the casual level, to be of questionable quantitative significance. Arithmetically, this case can be obtained from the present model by setting the elasticities of investment with respect to expected yields equal to zero. If $\alpha_1 = \alpha_2 = 0$ then, from (9.13), $\pi_3 = 0$ and (see Appendix C) $\sigma_k^2 = \sigma_{km} = 0$. For all markets and all t, $k_t(z) = k_t = \hat{k}_t = \pi_{10}/(1 - \pi_1)$.

The functions A_1 and A_2 are found equal to γ/π_4 and $\rho(1 - \gamma)$, respectively, where

$$\gamma = \frac{\sigma_m^2 + \sigma^2}{\sigma_m^2 + \sigma^2 + \sigma_\theta^2}.$$

Then, from (9.14),

$$\pi_4 = \frac{1 + \beta_2\gamma}{1 + \beta_2 - \beta_1\rho(1 - \gamma)}. \tag{11.1}$$

Letting σ_θ^2 range from 0 to infinity, the price response π_4 ranges from the high value of unity to a low value of $(1 + \beta_2 - \rho\beta_1)^{-1}$. In economic terms, as the fraction of demand variation due to aggregate nominal disturbances tends to unity, equilibrium prices tend to move in proportion to demand shifts. As this occurs, the output response tends to zero. Their ratio (the slope of the Phillips curve) tends to infinity.

[15] This is essentially a parametric version of the model in Lucas (1972), except that in the present version, monetary changes are perceived with a distributed (rather than a fixed one-period) lag. Setting $\sigma_m^2 = 0$ gives an exact counterpart to the model of Lucas (1972). A comparison of the two gives a good idea of the costs and benefits of working with parametrically specified demand functions rather than with preference functions of agents (see n. 4 above).

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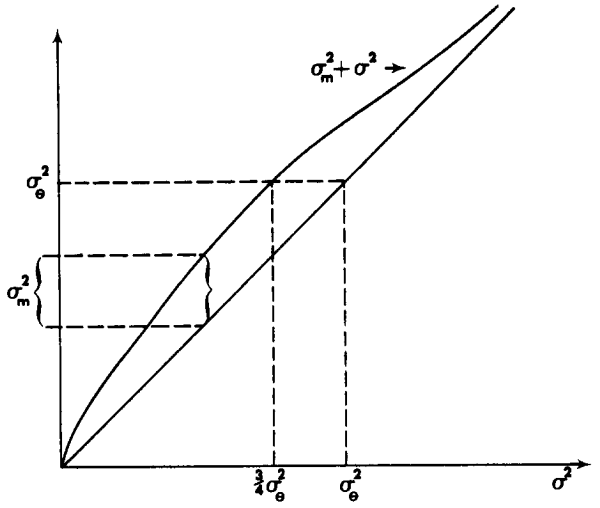


FIG. 1

It remains to determine σ_m^2 as a function of σ^2. For the case under consideration, (8.6) takes the form

$$\sigma_m^2 = (1-\gamma)^2(\sigma_m^2+\sigma^2) = \left[\frac{\sigma_\theta^2}{\sigma_m^2+\sigma^2+\sigma_\theta^2}\right]^2(\sigma_m^2+\sigma^2). \quad (11.2)$$

The solution for σ_m^2 and $\sigma_m^2+\sigma^2$ as functions of σ^2 are diagrammed in figure 1; σ_m^2 is zero when $\sigma^2 = 0$, with a derivative approaching $+\infty$; it reaches a maximum of $\frac{1}{4}\sigma_\theta^2$ when $\sigma^2 = \frac{3}{4}\sigma_\theta^2$; it tends to 0 as $\sigma^2 -> \infty$. The behavior of $\sigma_m^2+\sigma^2$ is as shown. The coefficient γ increases from 0 to 1 as σ^2 increases from 0; it equals $\frac{1}{2}$ when $\sigma^2 = \frac{3}{4}\sigma_\theta^2$.

The variance σ_m^2 is not, of course, the variance of the money supply (which *has* no stationary value when m_t follows a random walk). It is the average squared value of $m_t - \hat{m}_t$: the difference between the actual money supply and the level perceived, on average, by agents. When the monetary shock is small (σ^2 near zero) this error is small, since past information is a reliable guide to the present state. When σ^2 is very large, σ_m^2 is again small, since contemporaneous price movements provide an excellent indicator of movements in $m_t + x_t$. The error is greatest when σ^2 is of the same order of magnitude as σ_θ^2, so that monetary noise is sufficient to be economically interesting yet small enough to be confounded by agents with relative demand movements.

To obtain the dynamic behavior implied by these solution values, rewrite (9.10) (or [8.5]) as

$$m_{t+1} - \hat{m}_{t+1} = (1-\gamma)(m_t - \hat{m}_t + x_t - \mu), \quad (11.3)$$

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which is implied by the solution found above. Given exogenous money movements as assumed in (4.2), (11.3) describes the way agents' beliefs about the state of the economy move through time relative to the motion of the actual state.

To get a more concrete idea of the kind of "cycle" implied by this solution, it is useful to simulate the response to a single once-and-for-all demand shock (even though the occurrence of such a pattern has been assumed to have zero probability). Imagine an initial situation in which perceived and actual states are equal: $\hat{m}_0 = m_0$. There is an initial shock to demand: $x_0 - \mu = S$. Thereafter, money grows smoothly at its average expansion rate: $x_t - \mu = 0, t > 1$. From (11.3), agents will initially underestimate the true stock of money but will "catch on" at an exponential rate through time:

$$m_t - \hat{m}_t = (1 - \gamma)^t S, \; t \geq 1. \tag{11.4}$$

From (9.12), specialized to this case, the initial shock will induce a price increase above what had been expected in period 0 in the amount $\pi_4 S$. Prices will continue to stay above expectations due to lagged adjustments in $\hat{m}_t$ but by an exponentially decreasing amount. To be exact,

$$p_t - \hat{p}_t = \pi_4(1 - \gamma)^t S, \qquad t \geq 0.$$

This motion will not be affected by changes in the *average* monetary growth rate, although, of course, the path of actual prices will be.

The movements in flow variables—output, employment, and consumption—which parallel these price movements can be inferred from the income-expenditure identity (2.1), the link between monetary expansion and government spending, (2.5), and assumptions about households' preferences for labor supplied and goods consumed. For the latter, assume purely for simplicity that consumption does not vary over the cycle, so that fluctuations in government purchases are absorbed by employment fluctuations. In the case under discussion, capital and investment are also constant, so that (2.1) may then be solved for employment as a function of $g_t = \log (G_t)$. Expanding this function yields the approximation

$$n_t = \eta_0 + \eta_1 g_t, \tag{11.5}$$

where n_t is the log of employment. The elasticity η_1 is the average ratio of G to output divided by the elasticity of production f with respect to labor input (labor's share).

From (2.5) and (4.2), real government spending is in turn given by

$$g_t = x_t - p_t + m_t. \tag{11.6}$$

With capital fixed, (11.6) and (9.12) together yield

$$g_t = (1 - \pi_4)(x_t - \hat{m}_t + m_t) + \text{constant}. \tag{11.7}$$

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Then, combining (11.4), (11.5), and (11.7), the time path of the percentage deviations of employment from its normal level, resulting from a shock S, is given by

$$n_t - n^e = \eta_1(1 - \pi_4)(1 - \gamma)^t S, \qquad t = 0, 1, 2, \ldots. \tag{11.8}$$

Similarly, the expected yield on money will move in proportion to $m_t - \hat{m}_t$. The exact relationship is, from (6.10),

$$r_{mt} = \frac{(1 - \gamma)(1 + \beta_1\gamma)}{(1 + \beta_2\gamma)} \pi_4(1 - \gamma)^t S, \qquad t \geq 0, \tag{11.9}$$

where r_{mt} is the variable part of $r_{mt}(z)$ averaged over markets. The relationship of this pattern in r_{mt} to observed cyclical patterns in interest rates, which is by no means a simple issue, is discussed below (sec. 13).

One notes that the effects of an initial shock, in the purely monetary model, will *persist* but can never *cumulate*: the largest effect must come in the first period. To account for the observed gradual cyclical upswing, it appears that one must introduce systematic patterns in the shocks or modify the internal structure of the model.

12. Case 3: A Monetary Overinvestment Cycle

The preceding section exhibits a unique solution to the system (8.2), (8.3), (8.6), (9.11), and (9.12) for the case when $\alpha_1 = \alpha_2 = 0$. In this section, approximate solutions are developed and their properties discussed for the situation where α_1 and α_2 are small but positive. The details of this expansion are discussed in Appendix D; the main results are as follows.

For the accelerator coefficient π_3, one finds

$$\pi_3 = \left[\frac{1 - \gamma}{1 + \beta_2 - \beta_1\rho(1 - \gamma)}\right][\rho\alpha_1 - \alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\gamma], \tag{12.1}$$

where γ is the variance ratio defined in the preceding section. A sufficient condition for a positive accelerator effect, $\pi_3 > 0$, is $\rho\alpha_1 - \alpha_2 > 0$. Since $\alpha_1 > \alpha_2$, this will obtain if ρ is near 1. (If ρ were near zero, meaning that relative demand shifts were nearly transitory, one would not *expect* an accelerator effect, since new capital can only be installed with a one-period lag). As with the other real consequences of monetary shocks, the accelerator effect on investment is larger the smaller is the fraction γ of demand variance due to nominal shocks. In summary, to be induced to vary the investment rate, agents must (i) be responsive to perceived future relative returns ($\alpha_1 > 0$), (ii) be convinced that current relative demands are a good indicator of these future returns (ρ large), and (iii) be convinced that current price movements contain information on current relative demands (γ small).

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The effects of introducing a positive accelerator π_3 on the lagged *perceptions* of a monetary movement are easy to describe in words. The increased capacity due to an initial positive shock *retards* the upward adjustment of the price level to the new money introduced by the shock. The adjustment of expectations to the shock will then take place more slowly than the exponential pace described in (11.3). The details of these movements in perceptions are given in (9.9) and (9.10); expressions for the coefficients in these equations, valid for α_1 and α_2 small, are given in Appendix D.

The characteristic roots of this system are near α_3 and $1 - \gamma$, both in the unit interval, so that following a one-time shock, perceptions on both capital and money will return to "normal" in a nonoscillating fashion. The "cross effects" are probably both positive: underestimation of capacity ($\hat{k}_t - k_t < 0$) leads to underestimation of aggregate demand ($\hat{m}_{t+1} - m_{t+1} < 0$), and, similarly, $\hat{k}_{t+1} - k_{t+1}$ increases as $\hat{m}_t - m_t$ increases. In response to a pulse shock S, both $\hat{k}_t - k_t$ and $\hat{m}_t - m_t$ move proportionally to S. One (but not both) of these errors can continue to move in the same direction (that is, errors *can* cumulate) while the other decays. Eventually, both tend to zero.

Given the motion of the perception errors $\hat{k}_t - k_t$ and $\hat{m}_t - m_t$, as just discussed, the motion of *actual* capital stock k_t following an initial shock S is given by (9.11). The initial effect is $\pi_3 S$; subsequent effects in the same direction are contributed by the term $\pi_3(m_t - \hat{m}_t)$. Offsetting effects arise from $\hat{k}_t - k_t$. Since $\alpha_3 < \pi_1 < 1$, and since all three forcing terms tend to zero, k_t must eventually return to its normal level.

The consequences for employment of these movements in actual and perceived state variables can be obtained as in the preceding section. The presence of capital makes these calculations both more complicated and more interesting. Again, take consumption to be constant, "solve" (2.1) for the log of employment, and expand to obtain the analogue of (11.5):

$$n_t = \eta_0 + \eta_1 g_t + \eta_2(k_{t+1} - k_t) + \eta_3 k_t. \tag{12.2}$$

As before, the elasticity η_1 is the ratio of G to output divided by labor's share; η_2 is the average capital-output ratio divided by labor's share; and η_3 is capital's share divided by labor's share. Real spending, g_t, is obtained from (11.6) and (9.12):

$$g_t = \pi_2 k_t - (\beta_3\pi_4 - \pi_2)(\hat{k}_t - k_t) + (1 - \pi_4)(x_t - \hat{m}_t + m_t). \tag{12.3}$$

Combining (12.2) and (12.3) yields the time path of employment.

The direct "multiplier" effect on employment of a shock ($\eta_1 g_t$) works much as in the preceding section: there is an initial effect due to a movement in x_t followed by additional effects due to informational lags. The effect new to this section is the accelerator term $\eta_2(k_{t+1} - k_t)$, which can

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be relatively large even for small values of π_3. Further, since capital returns to normal, the term $\pi_2(k_{t+1} - k_t)$ must eventually make a *negative* contribution to employment, possibly driving employment below its normal level, even in the absence of a downward shock.

Movements in expected yields on both money and capital will, as in the preceding section, be procyclical.[16] These facts may be verified from (6.10) and (6.11), but the exact expressions need not be given here.

13. The Role of Interest Rates

The procyclical pattern of interest rate movements has perhaps attracted more theoretical and empirical attention in recent years than any other "stylized fact" concerning business cycles. The procyclical movement of the two expected yields, as shown in (11.9) and, for the general case, in (6.10) and (6.11), raises the hope that this fact too is accounted for by the model developed above. The question is worth examining, although it will turn out that a satisfactory answer remains beyond the scope of this paper.

Formally, the model above considers internal equity financing only, in contrast with the established convention that, in theories which consider one source of financing only, that one source should be bonds. This departure is obviously necessitated by the presence of uncertainty: the claim to an uncertain yield *cannot* be a single type of bond. One could *add* private bonds as an additional source of financing. If bond transactions were localized, as are goods transactions (that is, exchanges among agents in a single market), this would be easy to do, and one can conjecture that bond yields would move as the expected yields in (6.10) and (6.11). The interesting issue, however, is to examine the consequences of a single *economy-wide* market for some standardized kind of bond which would clear in an integral sense but not for each fixed market z. This modification would involve a major change in the information structure of the economy, since the equilibrium interest rate (or bond price) would depend only on *aggregate* state variables, and hence its value would convey to agents some aggregate information uncontaminated by local disturbances.

To see the effects of this, return to the inference problem solved by agents in section 6 and suppose that agents also observe the value of a known linear function of $\hat{k}_t - k_t$, $\hat{m}_t - m_t$, and x_t. The extreme consequence occurs when capital movements are unimportant, as in the purely

[16] Friedman (1971, p. 327) observes that cyclical variations in the average marginal productivity of capital are of slight quantitative importance, so that variation in the expected average yield on capital must play a minor cyclical role. This fact, as Friedman suggests elsewhere in the same article, is thus entirely consistent with an important cyclical role for average expected real yields.

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monetary model of section 11. In this case, the interest rate will convey the aggregate state of the economy *perfectly* to agents, eliminating the real part of the cycle altogether.[17] With an accelerator effect present, it seems likely that the existence of an economy-wide bond market would dampen cyclical movements but not eliminate them or alter their qualitative character. Without further analysis, however, the question remains open and, clearly, crucial.

The interest rate question illustrates an interesting analytical tension which must arise in any cycle theory based on incomplete information. On the one hand, it is easy to postulate agents and market institutions which ignore or foolishly waste information: the result is a theory which seriously understates agents' abilities to vary their decision rules with changes in the environment (such as, for example, the theory underlying the major econometric forecasting models). It is equally easy to postulate "efficient" securities markets which rapidly transmit *all* information to all traders: the result is a static general equilibrium model. To observe that one must avoid both extremes to understand the business cycle does not take one very far in discovering the correct "centrist" model, but it seems nonetheless an essential point of departure.

14. Remarks on Testability

The model described in section 12, and any other variant in this general class, ascribes values to all aggregate moments: the complete covariance function of the vector of observable variables. Since there are many more such sample moments than there are free parameters in the system, it is clear that the model has empirical content.

In the absence of an economic theory on the behavior of the shocks, one would in practice begin by describing the shocks stochastically by some ad hoc method, possibly using only past values of the series itself, possibly relating it to movements in other state variables. Then, based on these findings, one would need to redo the theory above (especially the inference problem in sec. 6), assuming that the same pattern in the disturbance is also known to traders. If, as seems likely, a fairly complicated pattern (say, three or four parameters) is required to describe the shocks, this will lead to *more*, not fewer, testable restrictions on the solution parameters.[18] In short, there appears to be little risk of the vacuity which mars so much of distributed lag econometrics.

In addition to aggregate predictions, the theory also "predicts" that

[17] Could not a given interest rate movement indicate ambiguously either a high x_t *or* a high $m_t - \hat{m}_t$? As a transient effect, yes, but not in the stationary distribution; see (8.5) with $\hat{k}_t - k_t = 0$.

[18] See Sargent's application of the principle of rationality to the Fisherian interest rate–inflation rate distributed lag (Sargent 1973*a*).

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deviations from average in the demand for individual products will be independent from product to product and through time. One may (as I did [Lucas 1972]) take these "predictions" metaphorically[19] (as one takes the prediction that all individuals are indistinguishable and live forever), but it is instructive to ask which covariance structures for individual demand shocks will lead to aggregate behavior "like" that described above and which will not. The answer seems to be that one needs each individual market shock to be expressable as a linear combination of a large number of roughly commensurate independent shocks (so that the law of large numbers applies as used above) plus a *single* shock common to all markets. This assumption, namely, that there exists some single random variable identifiable as aggregate demand, *is* testable and surely deserves systematic examination.

15. Remarks on Policy Implications

All aggregate output movements in the models studied above result from movements in a single monetary-fiscal shock to aggregate demand. Evidently, the key to any stabilization policy in such a setting would involve the elimination of any avoidable components of the variance of this shock. The present study, then, provides a rationalization for rules which smooth monetary policy, exactly as did the earlier studies of Lucas (1972), Sargent and Wallace (1973), and Barro (1975). Similarly, it rationalizes the analogous fiscal rule of continuous budget balancing and rules to stabilize the quantity of private money, such as larger reserve requirements for banks. Though it could be extended to do so, the present model sheds no light on the relative importance of monetary and fiscal effects, since all shocks, by assumption, involve both monetary and fiscal elements.

If, as seems likely in fact, some components of aggregate demand variance are unavoidable, the present model offers the additional possibility of stabilization by affecting the response characteristics of the private sector. For example, a fiscal stabilizer which reduced the parameters α_1 and α_2 (the elasticities of investment with respect to perceived, pretax rate-of-return changes) would convert the economy of section 12 to the more stable economy of section 11. This *feasibility* of stable but reactive stabilization policies will obtain, it would appear, in any model in which the effects of shocks persist through effects on capital accumulation.

In my view, the *desirability* of reactive policy rules is a more serious issue than is their feasibility. A tax policy which reduced the responsiveness of investment to aggregate demand changes would, as can be seen

[19] In his persuasive comment on Lucas (1973*a*), Vining (1974) utilizes a literal interpretation of this assumption to obtain a suggestive empirical test.

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from the analysis in the preceding sections, reduce the variance of aggregate output and employment. At the same time, such a policy would necessarily reduce the responsiveness of investment to relative demand shifts, retarding the movement of resources into the socially most desirable activities. Since the preferences and production possibilities in this model economy have not been made explicit, one cannot conclude with the presumption that the balancing achieved by the private sector in this model is efficient. On the other hand, it has not been necessary to introduce any of the standard types of "market failure" in order to account for the main features of the observed cycle.

16. Conclusion

This paper develops a theoretical example of a business cycle, that is, a model economy in which real output undergoes serially correlated movements about trend which are not explainable by movements in the availability of factors of production. The mechanism generating these movements involves unsystematic monetary-fiscal shocks, the effects of which are distributed through time due to informational lags and an accelerator effect. Associated with these output movements are (i) procyclical movements in prices, (ii) procyclical movements in the share of output devoted to investment, and (iii), in a somewhat limited sense, procyclical movements in nominal rates of interest.

This behavior is obtained under assumptions about expectations formation which seem suited to the study of a recurrent event: agents are well aware that the economy goes through recurrent "cycles" which distort perceived rates of return. On the other hand, the transitory nature of real investment opportunities forces them to balance the risk of incorrectly responding to spurious price signals against the risk of failing to respond to meaningful signals.

Appendix A

Equations (2.14) and (2.17) are two equations in the unknown parameters π_{11} and π_{21}. Solving them means essentially finding the roots of a quadratic, which can of course be done in several ways. A particularly convenient way is to add β_2 times (2.14) to α_2 times (2.17), obtaining the line

$$\pi_{21} = \frac{\beta_2 + (\beta_2\alpha_1 - \beta_1\alpha_2)\delta_1}{\alpha_2}\pi_{11} - \frac{\alpha_3\beta_2 + \alpha_2\beta_3}{\alpha_2}. \tag{A.1}$$

Rewrite (2.14) as

$$\pi_{21} = \frac{\alpha_3 - (1 + \alpha_1\delta_1)\pi_{11}}{\alpha_2(1 - \pi_{11})}. \tag{A.2}$$

The two solutions to (A.1) and (A.2) (that is, to [2.14] and [2.17]) are illustrated in figure A1. The relevant root economically is in the southeast quadrant; it satisfies the inequalities (2.19) and (2.20).

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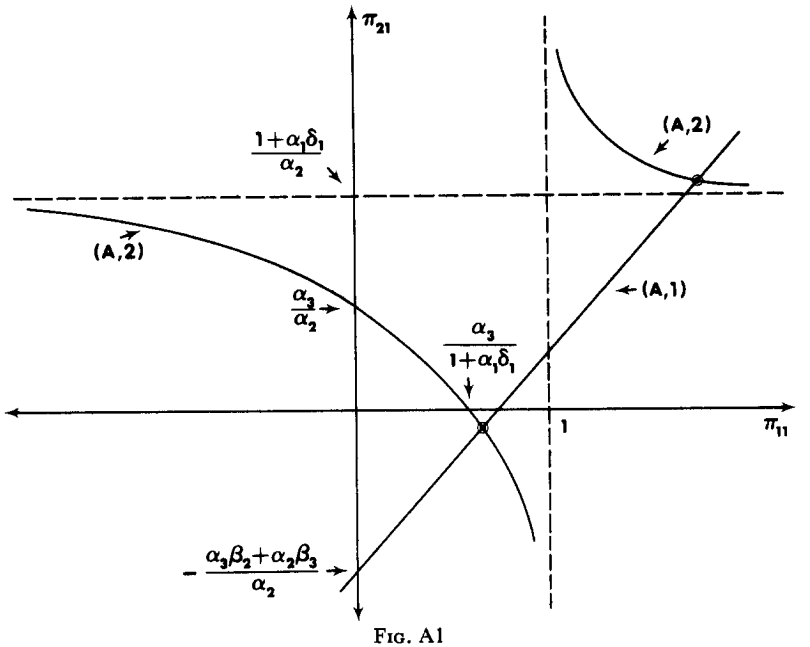


FIG. A1

To obtain $\pi_1 = \pi_{11} + \pi_{13}$ and $-\pi_2 = \pi_{21} + \pi_{23}$ (as in sec. 9), one proceeds in the same way. Adding (7.1) and (7.3) yields

$$\pi_1 = \alpha_3 - \alpha_2(-\pi_2)(1 - \pi_1). \tag{A.3}$$

Adding (7.6) and (7.8) gives

$$-\pi_2 = -\beta_3 - \beta_2(-\pi_2)(1 - \pi_1). \tag{A.4}$$

These are equivalent to (2.14) and (2.17), with $\delta_1 = 0$. Thus, $(\pi_1, -\pi_2)$ satisfy (A.1) and (A.2), with $\delta_1 = 0$, and their solution values appear in figure A1, with $\delta_1 = 0$. The inequalities (9.1) and (9.2) are easily verified.

Appendix B

Some expressions which arise in the text are

$$B_1 = \sigma_p^{-2}[\pi_{13}(\pi_{23}\sigma_k^2 + \pi_{24}\sigma_{mk}) + \pi_{14}(\pi_{23}\sigma_{mk} + \pi_{24}\sigma_m^2) + \pi_{15}\pi_{25}\sigma^2], \tag{B.1}$$

$$B_2 = \sigma_p^{-2}(\pi_{23}\sigma_{mk} + \pi_{24}\sigma_m^2 + \pi_{25}\sigma^2), \tag{B.2}$$

$$A_1 = (\pi_{21} + \pi_{23})B_1 + (\pi_{22} + \pi_{24})B_2, \tag{B.3}$$

$$A_2 = \sigma_p^{-2}[\pi_{13}\pi_{23}(\pi_{23}\sigma_u^2 + \pi_{25}\sigma u_\theta) + (\pi_{23}\pi_{15} + \rho\pi_{25})(\pi_{23}\sigma_{u\theta} + \pi_{25}\sigma_\theta^2)]. \tag{B.4}$$

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Define the quantities $M, D_1, \ldots, D_4$ by

$$M = \beta_3^2\sigma_k^2 - 2\beta_3\sigma_{mk} + \sigma_m^2 + \sigma^2 + \beta_3^2\sigma_u^2 - 2\beta_3\sigma_{u\theta} + \sigma_\theta^2, \quad \text{(B.5)}$$

$$D_1 = M^{-1}(\beta_3^2\sigma_k^2 - \beta_3\sigma_{mk}), \quad \text{(B.6)}$$

$$D_2 = M^{-1}(-\beta_3\sigma_{mk} + \sigma_m^2 + \sigma^2), \quad \text{(B.7)}$$

$$D_3 = M^{-1}(\beta_3^2\sigma_u^2 - \beta_3\sigma_{u\theta}), \quad \text{(B.8)}$$

$$D_4 = M^{-1}(-\beta_3\sigma_{u\theta} + \sigma_\theta^2). \quad \text{(B.9)}$$

Then, after the elimination of parameters described in section 9, (B.1)–(B.4) can be rewritten

$$B_1 = -\frac{\alpha_3}{\beta_3\pi_4}D_1 + \frac{\pi_3}{\pi_4}(D_1 + D_2), \quad \text{(B.10)}$$

$$B_2 = \frac{1}{\pi_4}D_2, \quad \text{(B.11)}$$

$$A_1 = \pi_2\frac{\alpha_3 - \beta_3\pi_3}{\beta_3\pi_4}D_1 + \frac{1 - \pi_2\pi_3}{\pi_4}D_2, \quad \text{(B.12)}$$

$$A_2 = (\alpha_3 - \beta_3\pi_3)D_3 + (\rho - \beta_3\pi_3)D_4. \quad \text{(B.13)}$$

To obtain the matrix K_1 used in (8.6), first abbreviate the coefficients of (8.4) and (8.5) by

$$C_{11} = \pi_{13} - \pi_{23}B_1 = \alpha_3 - \beta_3\pi_3 + \beta_3\pi_4B_1,$$

$$C_{12} = \pi_{14} - \pi_{24}B_1 = \pi_{15} - \pi_{25}B_1 = \pi_3 - \pi_4B_1,$$

$$C_{21} = -\pi_{23}B_2 = \beta_3\pi_4B_2,$$

$$C_{22} = 1 - \pi_{24}B_2 = 1 - \pi_{25}B_2 = 1 - \pi_4B_2.$$

Then

$$K_1 = \begin{pmatrix} C_{11}^2 & 2C_{11}C_{12} & C_{12}^2 \\ C_{11}C_{21} & C_{11}C_{22} + C_{12}C_{21} & C_{12}C_{22} \\ C_{21}^2 & 2C_{21}C_{22} & C_{22}^2 \end{pmatrix}. \quad \text{(B.14)}$$

Provided the matrix (C_{ij}) is stable (as required by the definition of a solution used here), the process (8.4)–(8.5) has a unique stationary covariance matrix. Since this covariance matrix is also a solution to (8.6) (and vice versa), it follows that (8.6) has a unique solution, or that $K_1 - I$ is a nonsingular matrix.

Appendix C

For the case $\sigma_\varepsilon^2 = 0$, $\sigma_u^2 = \sigma_{u\theta}^2 = \sigma_\theta^2 = 0$ from (8.2) and (8.3). Then, from (B.8)–(B.9), $D_3 = D_4 = 0$ so that from (B.13) $A_2 = 0$. It also follows that $C_{11} = (\alpha_3/\beta_3)C_{21}$ and $C_{12} = (\alpha_3/\beta_3)C_{22}$, so that by direct inspection of (8.4) and (8.5) one sees that $\sigma_k^2 = \alpha_3^2\sigma_m^2$ and $\sigma_{km} = \alpha_3\sigma_m^2$. These facts permit the calculation of M, D_1, and D_2 as functions of the two variances σ_m^2 and σ^2. Inserting these into the third equation of (8.6), one obtains a cubic in the unknown σ_m^2. The root zero occurs twice; the third root is negative. Thus, the unique solution of (8.6) is $\sigma_k^2 = \sigma_{km} = \sigma_m^2 = 0$.

With these variances, $D_1 = 0$ and $D_2 = 1$. Inserting these values into the

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expression for A_1 from (B.12), one finds that $\pi_3 = 0$ and $\pi_4 = 1$ is the unique solution to (9.13) and (9.14).

For the case $\alpha_1 = \alpha_2 = \pi_3 = 0$, use (8.6) to solve for σ_k^2 and σ_{km} as functions of $\sigma_m^2 + \sigma^2$. Evidently, (0, 0) is a solution, since in this case $D_1 = 0$. If $D_1 = 0$, there is no other solution. If $D_1 \neq 0$, one finds that $\sigma_k^2 < 0$, an impossibility. The solution given in the text is thus unique.

Appendix D

Equations (8.2), (8.3), (8.6), (9.11), and (9.12) are seven equations in the unknown reduced-form parameters $(\pi_3, \pi_4, \sigma_k^2, \sigma_{mk}, \sigma_m^2, \sigma_u^2, \sigma_{u\theta})$. In section 11 it was found that $(0, \bar{\pi}_4, 0, 0, \bar{\sigma}_m^2, 0, 0)$, with $\bar{\pi}_4$ given in (11.1) and $\bar{\sigma}_m^2$ by (11.2), is the unique solution when $\alpha_1 = \alpha_2 = 0$. Let $\alpha_2 = \xi\alpha_1$, where $\xi\varepsilon(0, 1)$ is a constant. Then the implicit function theorem implies that for α_1 sufficiently small, a differentiable solution exists. This solution can be approximated for α_1 small by expanding $(\pi_3, \pi_4, \sigma_k^2, \sigma_{mk}, \sigma_m^2, \sigma_u^2, \sigma_{u\theta})$ about the point $(0, \bar{\pi}_4, 0, 0, \bar{\sigma}_m^2, 0, 0)$.

Carrying out this expansion, (12.1) is immediate from (9.3) and (11.1). The approximate coefficients C_{ij} (see Appendix B) of the difference equations (9.9) and (9.10) are readily, if tediously, obtained in the same way. The expressions given below are obtained by expanding in α_1 and by discarding terms involving powers of γ of 2 or higher. The full expressions are not difficult to obtain, but there is little to be gained by repeating them here:

$$C_{11} = \alpha_3 - \beta_3(1 - \gamma)\pi_3 + \alpha_3 \frac{\gamma}{1 - \pi_3}\alpha_1,$$

$$C_{12} = (1 - \gamma)\pi_3 - \frac{\alpha_3\gamma}{1 - \alpha_3}\alpha_1,$$

$$C_{21} = \beta_3\gamma(1 - \frac{\beta_3\alpha_1}{1 - \alpha_3} - 2\frac{\beta_3\rho}{1 - \alpha_3}\pi_3),$$

$$C_{22} = 1 - \gamma + \beta_3 \frac{\gamma\alpha_1}{1 - \alpha_3} + 2\beta_3 \frac{\rho\gamma}{1 - \alpha_3}\pi_3.$$

For α_1 (and hence π_3) small, the roots of (C_{ij}) are seen to be near α_3 and $1 - \gamma$ (the two diagonal elements). C_{21} will be positive, as asserted in the text. C_{12}, treated also as positive in the text, can in fact have either sign, even for α_1 small. For γ small, it will be positive.

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